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HOMOGENEOUS PRODUCTION FUNCTION

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Abstract: **Almost all economic theories presuppose a production function, either on the firm level or the aggregate level. In this sense the production function is one of the key concepts of mainstream neoclassical theories. While there are many different production functions, only certain kinds of production functions are homogeneous. In general, they are multiplicative rather than additive although a few exceptions exist. Homogeneous production functions consist of a broad array of functions with a special characteristic. This paper therefore explores or examines production function and its features, homogenous production and its properties with policy application of elasticity of substitution. Constant elasticity of substitution was also considered. Despite the fact that the CES production function appears to be a very complicated function, it is a useful pedagogical tool in that it can be used to illustrate what happens to the shape of a series of isoquants as the elasticity of substitution changes.**

Keywords: **production function, homogenous production function, CES production function.**

1. INTRODUCTION

1.0 PRODUCTION FUNCTION: MEANING, DEFINITIONS AND FEATURES

Production is the result of co-operation of four factors of production viz., land, labour, capital and organization. This is evident from the fact that no single commodity can be produced without the help of any one of these four factors of production.

Therefore, the producer combines all the four factors of production in a technical proportion. The aim of the producer is to maximize his profit. For this sake, he decides to maximize the production at minimum cost by means of the best combination of factors of production.

The producer secures the best combination by applying the principles of equi-marginal returns and substitution. According to the principle of equi-marginal returns, any producer can have maximum production only when the marginal returns of all the factors of production are equal to one another. For instance, when the marginal product of the land is equal to that of labour, capital and organisation, the production becomes maximum.

It is well known that the production function is one of the key concepts of mainstream neoclassical theories, with a lot of applications not only in microeconomics and macroeconomics but also in various fields, like biology Adinya, Offem, and Ikpi (2011)), educational management, Cooper and Cohn (1997) and engineering, Boussard (2011). Roughly speaking, the production functions are the mathematical formalization of the relationship between the output of a firm/industry/economy and the inputs that have been used in obtaining it. In simple words, production function refers to the functional relationship between the quantity of a good produced (output) and factors of production (inputs).

In this way, production function reflects how much output we can expect if we have so much of labour and so much of capital as well as of labour etc. In other words, we can say that production function is an indicator of the physical relationship between the inputs and output of a firm. It is generally believed that the first economist to algebraically formulate the relationship between input and output was Philip Wicksteed (1984)

The reason behind physical relationship is that money prices do not appear in it. However, here one thing that becomes most important to quote is that like demand function a production function is for a definite period.

It shows the flow of inputs resulting into a flow of output during some time. The production function of a firm depends on the state of technology. With every development in technology the production function of the firm undergoes a change.

The new production function brought about by developing technology displays same inputs and more output or the same output with lesser inputs. Sometimes a new production function of the firm may be adverse as it takes more inputs to produce the same output.

Mathematically, such a basic relationship between inputs and outputs may be expressed as:

Q = f (L, C, N)

Where Q = Quantity of output

L = Labour

C = Capital

N = Land.

Hence, the level of output (Q), depends on the quantities of different inputs (L, C, N) available to the firm. In the simplest case, where there are only two inputs, labour (L) and capital (C) and one output (Q), the production function becomes.

Q = f (L, C)

1.1 Features of Production Function

The following are the main features of production function:

1.1.1. Substitutability: The factors of production or inputs are substitutes of one another which make it possible to vary the total output by changing the quantity of one or a few inputs, while the quantities of all other inputs are held constant. It is the substitutability of the factors of production that gives rise to the laws of variable proportions.

1.1.2. Complementarity: The factors of production are also complementary to one another, that is, the two or more inputs are to be used together as nothing will be produced if the quantity of either of the inputs used in the production process is zero. The principles of returns to scale is another manifestation of complementarity of inputs as it reveals that the quantity of all inputs are to be increased simultaneously in order to attain a higher scale of total output.

1.1.3. Specificity: It reveals that the inputs are specific to the production of a particular product. Machines and equipment’s, specialized workers and raw materials are a few examples of the specificity of factors of production. The specificity may not be complete as factors may be used for production of other commodities too. This reveals that in the production process none of the factors can be ignored and in some cases ignorance to even slightest extent is not possible if the factors are perfectly specific.

Production involves time; hence, the way the inputs are combined is determined to a large extent by the time period under consideration. The greater the time period, the greater the freedom the producer has to vary the quantities of various inputs used in the production process.

In the production function, variation in total output by varying the quantities of all inputs is possible only in the long run whereas the variation in total output by varying the quantity of single input may be possible even in the short run.

1.2 HOMOGENOUS PRODUCTION FUNCTION

The terms *economy* or *diseconomy* of scale can be confusing to interpret. Some economists define the terms with reference to a particular class of production functions, known as homogeneous production functions. Homogeneous production functions consist of a broad array of functions with a special characteristic. A production function is said to be homogeneous of degree n if when each input is multiplied by some number t, output increases by the factor t^n. That is if

$$f(tx_1, x_2) = t^n f(x_1 x_2)$$

Where n is a constant and t is any positive real number

Assuming that the time period is sufficiently long such that all inputs can be treated as variables and are included in the production function, n, the degree of homogeneity refers to the returns to scale. Homogeneous production functions are frequently used by agricultural economists to represent a variety of transformations between agricultural inputs and products. A function homogeneous of degree 1 is said to have constant returns to scale i. e output increases by the same proportion with the input combination under consideration. A function homogeneous of a degree greater than 1 is said to have increasing returns to scale or economies of scale i.e., output increases by a greater proportion with the input combination under consideration. A function homogeneous of degree less than 1 is said to have diminishing returns to scale or diseconomies of scale, meaning that output increases by a small proportion with the input combination under consideration.

While there are many different production functions, only certain kinds of production functions are homogeneous. In general, they are multiplicative rather than additive although a few exceptions exist.

The production function

$$y = Ax_1^{0.5}x_2^{0.5} \qquad (1)$$

is homogeneous of degree 1. Multiply x_1 and x_2 by t to get

$$A(tx_1)^{0.5}(tx_2)^{0.5} = tAx_1^{0.5}x_2^{0.5} \qquad (2)$$
$$= t^1 y$$

Thus, the function in equation (1) exhibits constant returns to scale without any economies or diseconomies.

The production function

$$y = Ax_1^{0.5}x_2^{0.8} \qquad (3)$$

is homogeneous of degree 1.3. Multiply x_1 and x_2 by t to get

$$A(tx_1)^{0.5}(tx_2)^{0.8} = t^{1.3}Ax_1^{0.5}x_2^{0.8} \qquad (4)$$
$$= t^{1.3}y$$

Thus increasing returns to scale and economies of scale exist.

The production function

$$y = Ax_1^{0.5}x_2^{0.3} \qquad (5)$$

is homogeneous of degree 0.8. Multiply x_1 and x_2 by t to get

$$A(tx_1)^{0.5}(tx_2)^{0.3} = t^{0.8}Ax_1^{0.5}x_2^{0.3} \qquad (6)$$
$$= t^{0.8}y$$

Thus decreasing returns to scale and diseconomies of scale exist. For multiplicative functions of the general form

$$y = Ax_1^{\alpha}x_2^{\beta} \qquad (7)$$

the degree of homogeneity can be determined by summing the parameters α and β.

Homogeneous production functions possess a unique characteristic. A line of constant slope drawn in factor-factor space will represent a proportionate change in the use of the inputs represented on the axes. For homogeneous functions, any line of constant slope drawn from the origin will connect all points on the isoquant map with equal slopes. In other words, any isocline has a constant slope for a homogeneous function.

Since an expansion path is a specific isocline with a slope -*v*1/*v*2, any homogeneous function will have an expansion path with a constant slope. (This characteristic is also true of a broader class of production functions, called *homothetic production functions*, which include homogeneous production functions as a special case.) For a homogeneous production function and fixed factor prices, movement along an expansion path, or, for that matter, movement along any isocline represents a proportionate change in the use of the inputs. For homogeneous production functions, if all inputs are included, movement along any isocline represents a change in the scale of an operation. One the most widely used homogenous production functions is the Cobb-Douglas production function here the function is homogenous of degree one and the MPs of X_1 and X_2 are homogenous of degree zero; i. e., they remain unchanged for proportionate changes of both inputs.

Examples of linearly homogeneous production functions are the Cobb-Douglas production function and the constant elas-ticity of substitution (CES) production function

1.2.1 Expansion Path of a Linear Homogenous Production Function:

Whether expansion path is linear or non-linear depends on the nature of technology involved in the production function. An important property of a linear homogeneous production function is that its expansion path is straight line from the origin as shown in figure 1. As we all know that expansion path represents optimal factor combinations as firm expands its output, given the prices of factors. At an optimal factor combination, $MRTS_{LK}$ is equal to factor price ratio ($MRTS_{LK}$ = w/r).

Since the factor prices remain constant along an expansion path, this implies that $MRTS_{LK}$ will also remain constant.

Now, expansion path being a straight line from the origin implies that factor ratio (K/L) remains the same throughout on the expansion path. To prove that expansion path of a linear homogenous production function is a straight line from the origin we take Cobb-Douglas production function

($Q = AK^{-1/2} L^{1/2}$) which is an important example of homogenous production function of the first degree.

Take Cobb-Douglas production function,

$$Q = AK^{1/2} L^{1/2}$$

Differentiating it with respect to labour

$$\frac{\partial Q}{\partial L} = \frac{1}{2} AK^{1/2} L^{-1/2} \qquad ...(1)$$

Differentiating it with respect to capital,

$$\frac{\partial Q}{\partial K} = \frac{1}{2} AK^{-1/2} L^{1/2} \qquad ...(2)$$

Dividing (1) by (2) we have

$$\frac{MP_L}{MP_K} = \frac{\partial Q}{\partial L} \Big/ \frac{\partial Q}{\partial K} = \frac{\frac{1}{2} AK^{\frac{1}{2}} L^{\frac{-1}{2}}}{\frac{1}{2} AK^{\frac{-1}{2}} L^{\frac{1}{2}}} = \frac{K}{L}$$

$$\text{or} \quad MRTS_{LK} = \frac{MP_L}{MP_K} = \frac{K}{L}$$

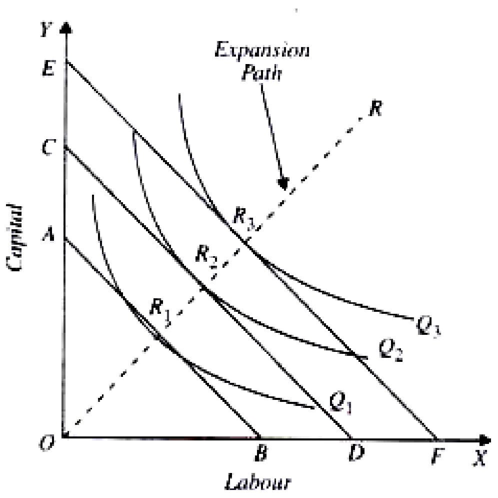

Fig 1: Graph showing that Expansion path of a linear homogeneous production

function is a straight line from the origin Thus, $MRTS_{LK}$ in the given linear homogenous Cobb-Douglas production function is equal to K/L. As explained above, at optimal factor combinations on the expansion path, $MRTS_{LK}$ equals w/r and, therefore, given the factor prices, $MRTS_{LK}$ remains constant along an expansion path. Thus K/L which is equal $MRTS_{LK}$ in a linear homogenous Cobb-Douglas production will remain constant.

Constant factor ratio K/L along the expansion path implies that it is a straight line from the origin.

1.3 POLICY APPLICATIONS OF THE ELASTICITY OF SUBSTITUTION

The elasticity-of-substitution concept has important applications to key issues linked to agricultural production. The liquid fuels energy crisis provides an illustration of the importance of the concept. Of concern is the extent to which other inputs can be substituted for liquid fuels energy in agricultural production. An example might be the potential substitutability between farm labor, farm tractors, and machinery and liquid fuels.

Agriculture in most foreign countries has become increasingly mechanized. Hence tractors and machinery can and do substitute for farm labor. This suggests that the elasticity of substitution is comparatively high between human labor and farm tractors and machinery. Massive changes in the mix of inputs required to produce agricultural products would not have taken place without clear economic signals. These economic signals are the relative prices for tractors and machinery and the fuel required to run versus farm labor. Farmers often complain about the prices for tractors and other farm machinery, but changes in the mix of inputs toward tractors and farm machinery would not have taken had it not been economic. Farmers look for the point of least-cost-combination today, much as they always have. If the relative proportions of each input do not change, or change very little in the face of changing relative input prices, then there is evidence to suggest that the elasticity of substitution between the inputs is nearly zero. However, when relative prices change and are accompanied by a change in the input mix, there is evidence in support of a positive elasticity of substitution.

1.3.1 CONSTANT ELASTICITY OF SUBSTITUTION (CES) PRODUCTION FUNCTION

A production function which belongs to the CES class has two major characteristics; (1) it is homogenous of degree 1, and (2) it has a constant elasticity of substitution. Production functions which lack one or both of these characteristics do not belong to the CES class. Thus, all production functions in theses class satisfy criterion (2). However, criterion (1) is satisfied only for $\alpha + \beta = 1$, that is for the Cobb –Douglas function. Since the Cobb Douglas type of production function imposes an elasticity of substitution between input pairs of exactly 1, then if a Cobb Douglas type of production function were estimated, the elasticity of substitution between input pairs would be an assumption underlying the research rather than a result based on the evidence contained in the data. The problem with the Cobb-Douglas type of production function is widely known and is of particular interest to economists engaged in macro-oriented issues, such as the extent to which capital could substitute for labour within an economy. The study published by Arrow, Chenery, Menhas, and Solow "Capital Labour Substitution and Economic Efficiency" in 1961 was a landmark. The study might also be considered a remake of the 1928 effort by Cobb and Douglas without the assumption that the elasticity of substitution between capital

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and labour was 1. In the study the authors first introduced the constant elasticity of substitution (CES) production function. The CES production function had two principal features. First, the elasticity of substitution between the two inputs could be any number between zero and infinity. Second, for a given set of parameters, the elasticity of substitution was the same on any point along the isoquant, regardless of the ratio of input use at the point: hence the name *constant elasticity of substitution production functions*.

The CES production function is

$$y = A[\lambda x_1^{-\rho} + (1 - \lambda)x_2^{-\rho}]^{-1/\rho}$$

The CES appears to be a very complicated function. The developers of the CES no doubt started with the result that they wished to obtain, a constant elasticity of substitution that could assume any value between zero and infinity, and worked toward a functional form that was consistent with this result. The elasticity of substitution (*e*s) and the parameter ρ are closely related

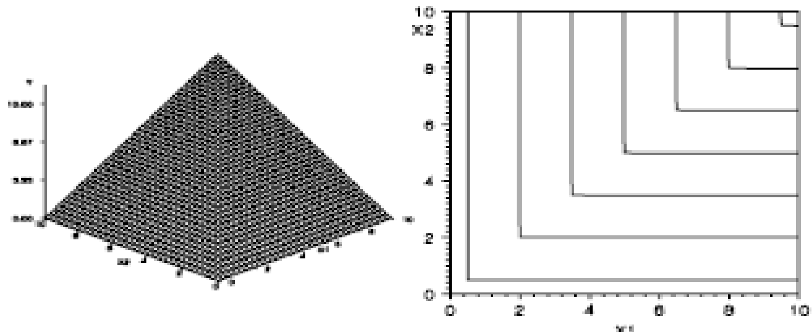
$$e_s = 1/(1 + \rho)$$

$$\rho = (1 - e_s)/e_s$$

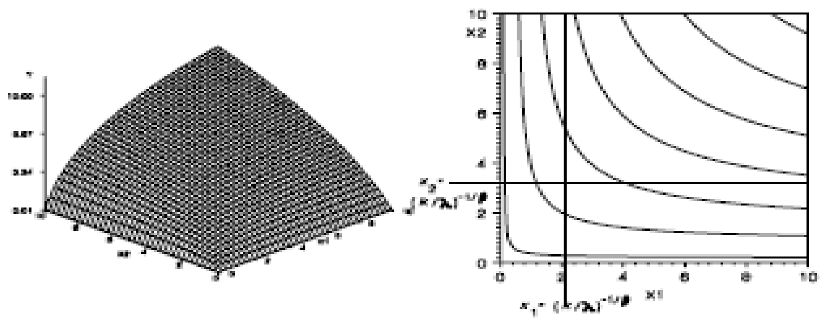
The authors retained the Cobb Douglas assumption of constant returns to scale in that $\lambda + (1 - \lambda) = 1$, but this assumption is not required.

In addition to having research application, the CES is a useful pedagogical tool in that it can be used to illustrate what happens to the shape of a series of isoquants as the elasticity of substitution changes. Henderson and Quandt (1980) suggest five possible cases. Figure 2 illustrates the production surfaces and corresponding isoquants generated under each of these cases.

Case 1: $\rho \to +\infty$, $e_s \to 0$. At the limit, substitution between input pairs is impossible and isoquants form a right angle. Diagrams A and B illustrate what happens as ρ becomes a rather large number. The shape of the production surface becomes like a pyramid. The production surface and isoquants illustrated in diagrams A and B was drawn with the assumption that ρ = 200.



A Case 1 ρ = 200 B Isoquants



C Case 2 ρ = 0.5 D Isoquants

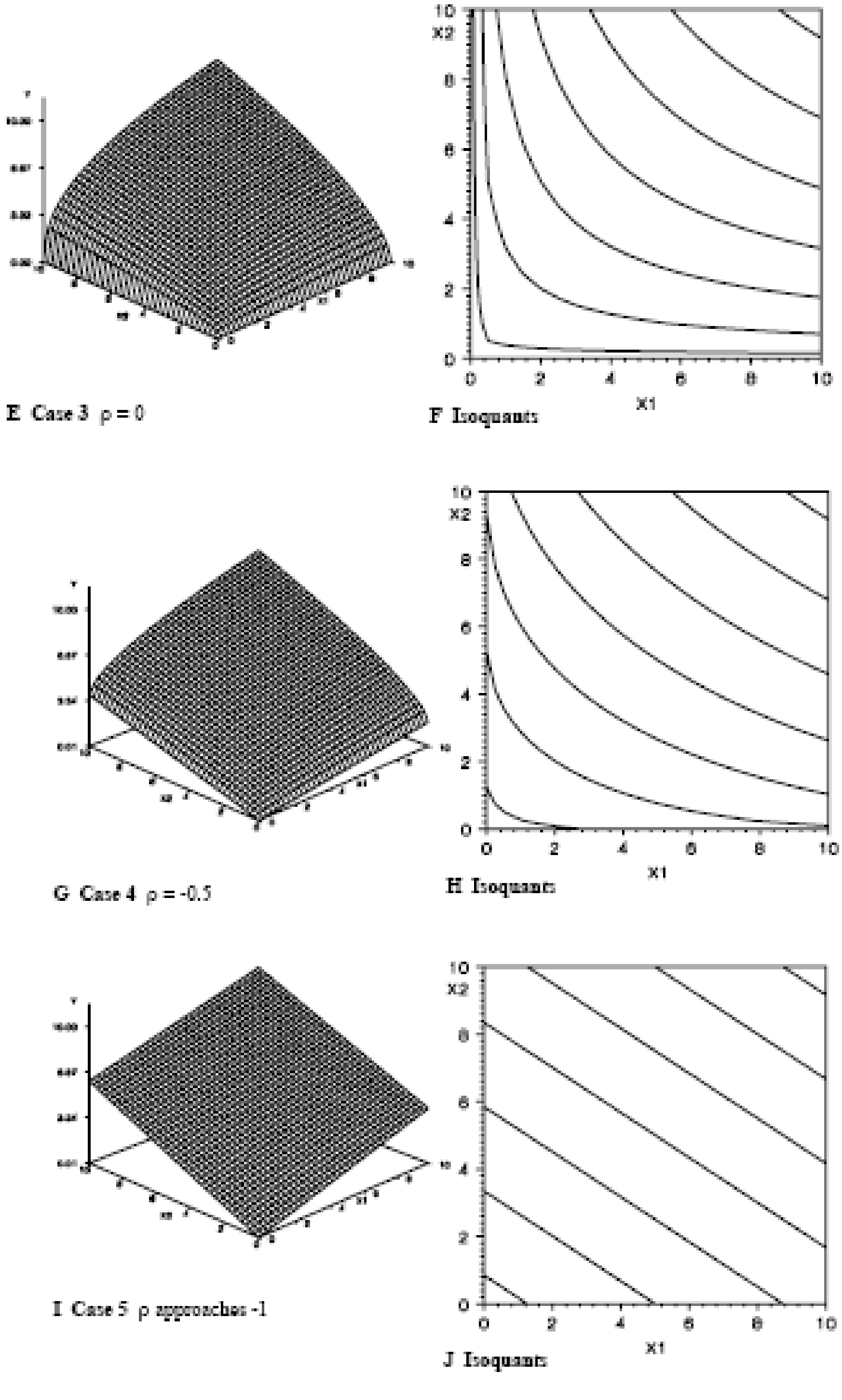


Figure 2: Production Surfaces and Isoquants for the *CES* Production Function under Varying Assumptions about ρ

Case 2: 0 < *e*s < 1ρ > 0. Inputs substitute for each other, but not very easily. The isoquants are asymptotic to some value for *x*1 and *x*2 rather than the axes. The vertical line is at $x_2 = (k/\lambda)^{-1/\rho}$, and the horizontal line is at $x1 = (k/(1- \lambda)^{-1/\rho}$. The number $k = (y/A)^{-\rho}$. The isoquants can be thought of as something in between the right angles in case 1 and those for a Cobb Douglas type function. Diagrams C and D illustrate the production surface and isoquant map when ρ = 0.5. The production surface is undistinguished and looks similar to that for the Cobb Douglas.

Case 3: *e*s = 1;ρ = 0. The CES becomes the Cobb Douglas illustrated in Diagrams E and F.

The proof of this requires the use of L'Hopital's Rule and can be found in Henderson and Quandt (1980).

Case 4: *e*s > 1; -1 < ρ < 0. Isoquants cut both axes. In diagram G and H, for ρ = -0.5, *e*s = 2, note the white area directly above the *x*1 and *x*2 axes. This suggests that output is possible in the absence of one of the two inputs.

Case 5: As *es*→→ +∞, ρ→ -1. At the limit the isoquants consist of lines of constant slope (with no curvature), and the production surface and isoquants are illustrated in diagram I and J. The CES reduces to the production function $y = \lambda x1 + (1 - \lambda)\, x_2$, and inputs substitute for each other in the fixed proportion λ/(1 - λ).

The CES had some important advantages over the Cobb Douglas production function in that the same general functional form could be used to represent a variety of substitution possibilities and corresponding isoquant patterns, but the function had two important disadvantages. Like the Cobb Douglas, for a given set of parameter values, only one stage of production could be represented, usually stage II for both inputs. This problem was not unrelated to the fact that the elasticity of substitution was the same everywhere along the isoquant. Isoquant patterns consisting of concentric rings or ovals were not allowed. The CES can be extended to allow for more than two inputs. However, there is but one parameter D in the multiple-input extensions. Thus only one elasticity of substitution value can be obtained from the production function, and this same value applies to all input pairs. For example, in agriculture, one might expect that the elasticity of substitution between chemicals and labour would differ markedly from the elasticity of substitution between fuel and tractors. But the CES would estimate the same elasticity of substitution between both input pairs. Despite its pedagogical charm for understanding the effects of changing elasticities of substitution on the shape of isoquants, the usefulness of the CES production function for serious research in agricultural economics in which more than two inputs were involved is limited.

1.4 KARUSH–KUHN–TUCKER CONDITIONS

In mathematical optimization, the Karush–Kuhn–Tucker (KKT) conditions, also known as the Kuhn–Tucker conditions, are first-order necessary conditions for a solution in nonlinear programming to be optimal, provided that some regularity conditions are satisfied. Allowing inequality constraints, the KKT approach to nonlinear programming generalizes the method of Lagrange multipliers, which allows only equality constraints. The system of equations and inequalities corresponding to the KKT conditions is usually not solved directly, except in the few special cases where a closed-form solution can be derived analytically. In general, many optimization algorithms can be interpreted as methods for numerically solving the KKT system of equations and inequalities. Boyd, Stephen, Vandenberghe, Lieven (2004).

The KKT conditions were originally named after Harold W. Kuhn, and Albert W. Tucker, who first published the conditions in 1951. Kuhn, Tucker, (1951). Later scholars discovered that the necessary conditions for this problem had been stated by William Karush in his master's thesis in 1939. Kjeldsen, Tinne Hoff (2000).

Often in mathematical economics the KKT approach is used in theoretical models in order to obtain qualitative results Chiang, Alpha (1984). For example, consider a firm that maximizes its sales revenue subject to a minimum profit constraint. Letting Q be the quantity of output produced (to be chosen), R (Q) be sales revenue with a positive first derivative and with a zero value at zero output, C (Q) be production costs with a positive first derivative and with a non-negative value at zero output, and G_{min} be the positive minimal acceptable level of profit, then the problem is a meaningful one if the revenue function levels off so it eventually is less steep than the cost function. The problem expressed in the previously given minimization form is

Minimize − R (Q)

Subject to G min ≤ R (Q) − C (Q)

Q ≥ 0,

and the KKT conditions are

$$\left(\frac{\mathrm{d}R}{\mathrm{d}Q}\right)(1+\mu) - \mu\left(\frac{\mathrm{d}C}{\mathrm{d}Q}\right) \le 0,$$

$$Q \ge 0,$$

$$Q\left[\left(\frac{\mathrm{d}R}{\mathrm{d}Q}\right)(1+\mu) - \mu\left(\frac{\mathrm{d}C}{\mathrm{d}Q}\right)\right] = 0,$$

$$R(Q) - C(Q) - G_{\min} \ge 0,$$

$$\mu \ge 0,$$

$$\mu[R(Q) - C(Q) - G_{\min}] = 0.$$

Since Q = 0 would violate the minimum profit constraint, we have Q > 0 and hence the third condition implies that the first condition holds with equality. Solving that equality gives

$$\frac{\mathrm{d}R}{\mathrm{d}Q} = \frac{\mu}{1+\mu}\left(\frac{\mathrm{d}C}{\mathrm{d}Q}\right)$$

Because it was given that dR/dQ and dC/dQ are strictly positive, this inequality along with the non-negativity condition on μ guarantees that μ is positive and so the revenue-maximizing firm operates at a level of output at which marginal revenue dR/dQ is less than marginal cost dC/dQ — a result that is of interest because it contrasts with the behavior of a profit maximizing firm, which operates at a level at which they are equal.

1.5 DUALITY IN PRODUCTION

Agricultural economists are perhaps most familiar with the concept of duality as it relates to linear programming models. Within a linear programming context, duality refers to the fact that any linear programming problem can be expressed either as a maximization problem or a corresponding minimization problem subject to appropriate constraints. The primal problem may be either maximization or a minimization problem. If the primal is a maximization problem, the corresponding dual will be a minimization problem, and, conversely, if the primal is a minimization problem, the corresponding dual will be a maximization problem.

The key characteristic of the dual relationship, as illustrated by a linear programming problem, is that all of the information about the solution to the primal can be obtained from the corresponding dual, and all of the information with respect to the solution of the dual can be obtained from the corresponding primal. Either the maximization or the minimization problem may be solved as the primal, and all information regarding the solution to the dual is obtained without resolving the problem.

Production functions have corresponding dual cost functions or perhaps correspondences. The term dual used in this context means that all of the information needed to obtain the corresponding cost function is contained in the production function, and, conversely, the cost function contains all of the information needed to derive the underlying production function.

To derive the production function from the cost function we shall consider a firm's isoquant defined by

$q^o = f(x_1,x_2)$ and the first order condition for cost minimization for this output is

$-d_{x2}/d_{x1}=r_1/r_2$ solving these equations for the input functions we have

$X_1=\psi\ (r_1/r_2,\ qo)$

$X_2=\psi\ (r_1/r_2,\ qo)$

Where x_1 and $x_2$are cost minimizing values expressed as functions of the ratio of the input prices and the prescribed output level. The partial derivatives of the cost function with respect to the input prices equal the cost minimizing values for the inputs.

Typically duality theorems states that

1. A concave production function yields a cost function homogeneous of degree one in input prices given specified regularity conditions.

2. A cost function homogeneous of degree one in input prices yields a concave production function given a specified regularity conditions

3. A cost function derived from a particular production function will in turn yield that production function.

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